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ORIGINAL ARTICLE

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A side sensitive group runs control chart for detecting shifts in the process mean

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Abstract Gadre and Rattihalli (2004) have introduced a 'Group Runs' (GR) control chart for detecting shifts in the process mean, which is an improvement on the synthetic control chart proposed by Wu and Spedding (2000). In this article, we develop the 'Side Sensitive Group Runs' (SSGR) chart to detect shifts in the process mean. The SSGR chart performs better than the Shewhart's $\overline{X}$ chart, the synthetic chart and the GR chart. In steady state also, its performance is better than the remaining three charts.

Keywords $\overline{X}$ chart · CRL chart · Synthetic chart · Average time to signal

1 Introduction

To monitor a change in the process mean μ_0, Shewhart's $\overline{X}$ chart is widely used in industries. Though this control chart detects large shifts in the process mean effectively, its performance is 'poor' in detecting small to moderate shifts. In such situations, 'Exponentially Weighted Moving Average' (EWMA) control chart proposed by Roberts (1959) or 'Cumulative Sum' (CUSUM) chart (Page, 1954) can be used. For further details of these charts one may refer to Lucas and Saccucci (1990),

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Montgomery (1996). However these two charts are 'inferior' to the Shewhart's $\overline{X}$ chart in detecting large shifts in the process mean.

Therefore in the last two decades, the control charts are developed by combining two such charts. For example, $\overline{X}$-EWMA control chart proposed by Albin et al. (1997). This chart is a combination of the $\overline{X}$ chart and the EWMA chart. Though the chart is efficient to detect large shifts in the process mean as compared to the EWMA chart, it is inefficient to detect small to moderate shifts.

The 'Conforming Run Length' (CRL) chart proposed by Bourke (1991) can be used to detect any type of shift in the process level. Bourke (1991) defined CRL as the number of non-defective units inspected between two successive defective units. This chart will indicate a shift in the process level, if CRL is less than L_{CRL}, the lower limit of the chart. In his review paper, Woodall (1997) has recommended the use of CRL charts in process control.

Wu and Spedding (2000) proposed the synthetic control chart for detecting small shifts in the process mean by combining Shewhart's $\overline{X}$ chart and the CRL chart (treating sample (group) as a unit). Let μ_0 be the target value of the mean and σ^2 be the known process variance. Let δ be the shift in the target value μ_0 of the process mean in terms of process standard deviation (σ). In synthetic control chart if $\overline{X} \notin (\mu_0 - k\sigma/\sqrt{n}, \mu_0 + k\sigma/\sqrt{n})$, the group is declared as non-conforming. Further, for $r \geq 1$, Wu and Spedding (2000) defined Y_r (the r^{th} group-based CRL) as the number of conforming groups inspected between $(r-1)^{\text{th}}$ (if one such exists) and the r^{th} non-conforming group including the r^{th} non-conforming group. Synthetic chart declares the process as out of control when for some $r \geq 1$, $Y_r \leq L_s$ (the lower control limit of the chart) for the first time. As an illustration, let $L_s = 3$, $Y_1 = 4$, $Y_2 = 5$, $Y_3 = 2$. In this case, the synthetic chart gives a signal after observing third non-conforming group. It is numerically illustrated that, to detect small to moderate shifts, the synthetic chart performs better than the $\overline{X}$ chart.

Davis and Woodall (2002) highlighted an important aspect of side sensitivity and have illustrated that the synthetic chart with side sensitivity performs better than the synthetic chart. In side sensitivity, a type of the shift along with the value of the CRL is taken into consideration. Davis and Woodall (2002) defined the rule for side sensitive synthetic control chart as 'Declare the process as out of control if two out of $(L_s + 1)$ group means fall outside the control limits of the chart'. They computed the ARL values for the side sensitive synthetic chart for various values of L, and have not computed optimum values of n and k as well.

It is to be noted that the event $(Y \leq L_s)$ may be due to a shift in the process mean (a correct signal) or may be due to the natural variability (a false alarm). Therefore, when such an event occurs, it is desirable to monitor a process further to identify cause for the event $(Y \leq L_s)$. Considering an above fact, Gadre and Rattihalli (2004) proposed the 'Group Runs' (GR) control chart, which is an extension of the synthetic control chart. GR chart declares the process as out of control, if Y_1 (the first value of group-based CRL) $\leq L_g$, (the lower control limit of the chart) or for some $r (\geq 2)$ Y_r and $Y_{(r+1)}$ are not exceeding L_g for the first time. It is illustrated that, to detect small to moderate shifts in the process mean, GR chart performs better than the $\overline{X}$ chart, the synthetic chart and the side sensitive synthetic chart.

In this article, we develop 'Side Sensitive Group Runs' (SSGR) chart for shift in the mean and find optimal values of the control parameters (n, k, L) of the chart.

If Y_1, the first group-based CRL of the GR chart is greater than L_g, the lower limit of the GR chart, and for some $r = 2, 3, \ldots$ we have the situation that $Y_r \leq L_g$ and $Y_{r+1} \leq L_g$, but r^{th} and $(r+1)^{\text{th}}$ non-conformed groups are having shifts on opposite side of the target, then it does not essentially suggest that there is a shift in the process mean, but it might be an indication of an increase in the process variance(σ^2, say). In SSGR chart, we declare the process as out of control, if $Y_1 \leq L_{\text{ssg}}$ (the lower limit of the chart) or for some r = 2, 3, ... we have the situation that $Y_r \leq L_{\text{ssg}}$ and $Y_{r+1} \leq L_{\text{ssg}}$ and the r^{th} and $(r+1)^{\text{th}}$ non-conformed groups are indicating shifts on the same side of a target value μ_0 of the process mean.

Remainder of the paper is organized as follows. The notations required to design the SSGR chart are explained in the following section. The SSGR chart is discussed and developed in the same section. Numerical illustrations and the comparison of the SSGR chart with the other matched compatible variable control charts is carried out in Section 3. In Section 4, steady state performance of the SSGR chart is compared with the other variable control charts. Concluding remarks are included in the last section.

2 The notations and the chart

2.1 Notations

The following are some notations required to develop the SSGR chart.

1. ATS(δ) : average number of units required by SSGR chart to detect a shift in process mean from μ_0 to $\mu_0 \pm \delta\sigma$;
2. δ_1: shift in the mean, the magnitude of which is considered large enough to seriously impair the quality of the product
   and
3. τ: the minimum required value of $(\text{ATS})_0$.

2.2 The SSGR (n, k, L) chart

SSGR chart to detect small shifts in the process mean is implemented as follows.

1. Observe n items in succession constituting a group.
2. Declare the group as conformed (non-conformed) according to the group mean falls within (outside) the limits $L_{\overline{X}/s} = \mu_0 - k\sigma/\sqrt{n}$ and $U_{\overline{X}/s} = \mu_0 + k\sigma/\sqrt{n}$ of the $\overline{X}$ sub chart.
3. Declare the process as out of control, if $Y_1 \leq L_{\text{ssg}}$, the lower limit of SSGR control chart or for some $r > 1$, $Y_r \leq L_{\text{ssg}}$, $Y_{r+1} \leq L_{\text{ssg}}$ for the first time and the related non-conformed group means lie on the same side of the target value μ_0.

When the process goes out of control, stop the process, take corrective actions to set the target value and then restart the process.

2.3 Design of SSGR chart

By designing the SSGR chart we mean, for given values of the input parameters (μ_0, δ_1, τ), searching optimal values of the control parameters (n, k, L) under certain suitable criterion. Here we compute these values by using the following ATS criterion.

$$\left.\begin{array}{r}\text{minimise ATS }(\delta_1)\\ \text{subject to the constraint}\\ \text{ATS}(0) \geq \tau.\end{array}\right\} \tag{1}$$

If N is the number of defective groups observed before declaring the process has gone out of control then

$$\text{ATS}(\delta) = nE\left(\sum_{r=1}^{N} Y_r\right). \tag{2}$$

Using Wald's identity, from (2), we have,

$$\text{ATS}(\delta) = nE(Y_r)E(N). \tag{3}$$

Let P be the probability that the group being non-conformed when the process mean is shifted from μ_0 to $\mu_0 \pm \delta\sigma$. If it is assumed that X has normal distribution, then,

$$P = P(\delta) = 1 - P\left\{L_{\overline{X}/S} < \overline{X} < U_{\overline{X}/S} | \overline{X} \sim N\left(\mu_0 + \delta\boldsymbol{\sigma}, \boldsymbol{\sigma}/\sqrt{n}\right)\right\}$$

Thus, we have,

$$P = 1 - \boldsymbol{\Phi}(k - \delta\sqrt{n}) + \Phi(-k - \delta\sqrt{n}). \tag{4}$$

Notice that, $Y_r (r = 1, 2, \ldots)$ are independent and identically distributed (i.i.d.) geometric random variables with mean $(1/P)$. Therefore,

$$\text{ATS}(\delta) = (n/P)\, E(N). \tag{5}$$

To find E(N), we use CRL, based Markov chain representation of the chart.

Davis and Woodall (2002) have discussed the Markov chain representation of the synthetic control chart to identify shifts in the process mean using $\overline{X}$-based procedure. Here, we discuss the Markov chain representation of SSGR chart, using states of the group-based CRL. To write a 'transition probability matrix' (t.p.m.) related to an absorbing state Markov chain representation of the SSGR chart, it is necessary to add a few notations.

Define,

$$A = P(Y_r \leq L) = 1 - (1 - P)^L \tag{6}$$

and

$$\alpha = (1 - \phi(k - \delta\sqrt{n}))/P. \tag{7}$$

Let '+' denote the event that $\{Y \le L_{\text{ssg}}$ and $\overline{X} > U_{\overline{X}/S}\}$, '−' be the event that $\{Y \le L_{\text{ssg}}$ and $\overline{X} < L_{\overline{X}/S}\}$ and 'm' be the event that $\{Y > L_{\text{ssg}}\}$. Further, 'Signal' indicates, the event of two successive Y's not exceeding L_{ssg} with group means of the related non-conformed groups lie on the same side of the target value. Here we consider '±' as a head start, which means $\{Y_0 \le L_{\text{ssg}}, \overline{X} < L_{\overline{X}/S}$ and $\overline{X} > U_{\overline{X}/S}\}$. This is an artificial situation, but useful as a head start. Then, t.p.m. corresponding to the SSGR chart can be given as below.

$$
\begin{array}{l}
 \\
\text{mm} \\
\text{m}+ \\
\text{m}- \\
+\text{m} \\
-\text{m} \\
+- \\
-+ \\
\text{m}\pm \\
\pm\text{m} \\
\text{Signal}
\end{array}
\begin{array}{cccccccccc}
\text{mm} & \text{m}+ & \text{m}- & +\text{m} & -\text{m} & +- & -+ & \text{m}\pm & \pm\text{m} & \text{Signal} \\
(1-A) & \alpha A & (1-\alpha)A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (1-A) & 0 & (1-\alpha)A & 0 & 0 & 0 & \alpha A \\
0 & 0 & 0 & 0 & (1-A) & 0 & \alpha A & 0 & 0 & (1-\alpha)A \\
(1-A) & \alpha A & (1-\alpha)A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(1-A) & \alpha A & (1-\alpha)A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & (1-A) & 0 & \alpha A & 0 & 0 & (1-\alpha)A \\
0 & 0 & 0 & (1-A) & 0 & (1-\alpha)A & 0 & 0 & 0 & \alpha A \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-A) & A \\
(1-A) & \alpha A & (1-\alpha)A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

In this case, the initial state is m±. Hence, taking the sum of the elements of eighth row of $(I - R)^{-1}$, it can be shown that,

$$E(N) = \left\{1 - \alpha(1-\alpha)A^2\right\} / \left\{A^2[1 + \alpha(1-\alpha)(A-2)]\right\} \tag{8}$$

Thus from equation (5), we have

$$\text{ATS}(\delta) = (\text{n/P}) \left\{1 - \alpha(1-\alpha)A^2\right\} / \left\{A^2[1 + \alpha(1-\alpha)(A-2)]\right\} \tag{9}$$

A three-step procedure of obtaining the control parameters n, k and L of the chart is similar to that discussed in Gadre and Rattihalli (2004). For the sake of completeness, in the following, we give a brief algorithm of obtaining the parameters of SSGR chart. Let the suffices '0' and '1', respectively, indicate that the related quantities are computed by substituting $\delta = 0$ and $\delta = \delta_1$.

Algorithm

1. Supply the values of μ_0, σ, δ_1 and τ.
2. As ATS_1 never exceeds ATS_0, assign the value τ to ATS_1 and m_1 (the minimum value of ATS_1 ever attained) and unity to the group size n.
3. If $n > m_1$, terminate the search procedure and print the results; else move to the next step.
4. Assign the value 0.01 to k.
5. If $k > 3$, move to the Step-13; else, move to the next step.
6. Compute P_0, P_1, α_0 and α_1 using the formulae (4) and (7).
7. Initialize L to 1.
8. If L exceeds 20,000, go to the Step-5 after increasing k by 0.01; else move to the next step.

9. Compute A_0, A_1, ATS_0 and ATS_1 using, (6) and (9).
10. If ATS_1 exceeds m_1, move to Step-8 after increasing the value of L by unity.
11. If ATS_0 is less than τ, move to Step-5 after increasing k by 0.01.
12. If $\text{ATS}_1 < m_1$, assign the value of ATS_1 to m_1, ATS_0 to m_0, n to n_{ssg}, k to k_{ssg}, and L to L_{ssg}, increase L by unity and move back to Step-8.
13. Increase n by unity and move back to Step-3.

In the following section, to compare the performance of SSGR chart with the Shewhart's $\overline{X}$ chart, the synthetic chart and the GR chart, we consider the sets of input parameters from Gadre and Rattihalli (2004). A Macro in Mat-Lab is developed to obtain the control parameters of SSGR chart for given input parameter values and is used for the following illustrations.

3 Numerical Illustrations

3.1 Example 1

Let $\mu_0 = 0, \sigma = 1, \delta_1 = 0.2$ and $\tau = 10{,}000$. For these input parameters, values of the control parameters for the Shewhart's $\overline{X}$ chart, the synthetic chart, the GR chart and the SSGR chart along with respective ATS_1 values are given in Table 1.

This example shows that, not only ATS_1 of SSGR chart for mean is less than ATS_1 of the other three charts, but also the group size n_{ssg} of the SSGR chart is less than the group sizes of the remaining three charts.

Another way of comparing the performance of the charts is to compare normalized $\text{ATS}(\delta)$ values of different charts (normalized 'with respect to' (w.r.t.) the GR chart) related to various values of δ. Table 2 enlists such values for the Shewhart's $\overline{X}$ chart, the synthetic chart and the SSGR chart. The values for GR chart are unity and hence are not included in the table. The values are computed by varying δ from 0 to 3.

The graphs of normalised $\text{ATS}(\delta)$ against δ values related to Table 2 are given in Fig 1. From these graphs, it is observed that, for $\delta > 0$, except for very small values of $\delta (\leq 0.01$ in this example),

$$\text{ATS}_{\text{ssg}}(\delta) < \text{ATS}_g(\delta) < \text{ATS}_s(\delta) < \text{ATS}_{\overline{X}}(\delta).$$

Thus, though values of the control parameters are computed for specific value δ_1 of δ, SSGR chart detects a shift of any size (except for very small values of δ) in the process mean earlier than the $\overline{X}$ chart, synthetic chart and GR chart.
For $\delta \leq 0.01$, we have,

$$\text{ATS}_{\text{ssg}}(\delta) > \text{ATS}_g(\delta) > \text{ATS}_s(\delta) > \text{ATS}_{\overline{X}}(\delta).$$

Table 1 Values of the Control parameters and $\text{ATS}(\delta_1)$ for various charts

Control chart	n	k	L	ATS_1
Shewhart's $\overline{X}$ chart	186	2.353445	–	288
Synthetic chart	102	1.938719	4	201
GR chart	98	1.594030	3	165
SSGR chart	89	1.52	3	152

Table 2 Normalised ATS values for three charts

δ	$ATS_{\overline{X}}$	ATS_S	ATS_{SSG}	δ	$ATS_{\overline{X}}$	ATS_S	ATS_{SSG}	δ	$ATS_{\overline{X}}$	ATS_S	ATS_{SSG}
0.00	1	1	1.0068	0.23	1.4895	1.1979	0.9327	0.46	1.8174	1.0752	0.9373
0.01	0.9862	0.9987	0.9886	0.24	1.5091	1.1877	0.9347	0.47	1.819	1.0748	0.9372
0.02	0.9552	0.9962	0.9441	0.25	1.5295	1.1771	0.9363	0.48	1.8202	1.0745	0.9371
0.03	0.9264	0.9956	0.893	0.26	1.5506	1.1665	0.9374	0.49	01.8212	1.0743	0.937
0.04	0.9122	0.9998	0.8501	0.27	1.5724	1.1561	0.9382	0.50	1.8218	1.0741	0.937
0.05	0.9158	1.0103	0.821	0.28	1.5947	1.1462	0.9388	0.51	1.8223	1.074	0.937
0.06	0.9349	1.0272	0.8052	0.29	1.6171	1.1369	0.9392	0.52	1.8227	1.0739	0.9369
0.07	0.9659	1.0492	0.8001	0.30	1.6394	1.1282	0.9394	0.53	1.8229	1.0738	0.9369
0.08	1.0049	1.0745	0.8029	0.31	1.6612	1.1203	0.9395	0.54	1.8231	1.0738	0.9369
0.09	1.0487	1.1014	0.8109	0.32	1.682	1.1133	0.9395	0.55	1.8233	1.0738	0.9369
0.10	1.0948	1.128	0.8221	0.33	1.7016	1.1069	0.9395	0.56	1.8233	1.0737	0.9369
0.11	1.141	1.153	0.8348	0.34	1.7198	1.1014	0.9393	0.57	1.8234	1.0737	0.9369
0.12	1.1859	1.1752	0.8481	0.35	1.7363	1.0966	0.9392	0.58	1.8234	1.0737	0.9369
0.13	1.2282	1.1941	0.8612	0.36	1.751	1.0924	0.939	0.59	1.8235	1.0737	0.9369
0.14	1.2672	1.209	0.8735	0.37	1.764	1.0889	0.9388	0.60	1.8235	1.0737	0.9368
0.15	1.3026	1.22	0.8848	0.38	1.7752	1.0859	0.9386	0.61	1.8235	1.0737	0.9368
0.16	1.3343	1.2271	0.8948	0.39	1.7847	1.0835	0.9383	0.62	1.8235	1.0737	0.9368
0.17	1.3626	1.2306	0.9036	0.40	1.7928	1.0814	0.9381	0.63	1.8235	1.0737	0.9368
0.18	1.3877	1.2308	0.9111	0.41	1.7994	1.0797	0.9379	0.64	1.8235	1.0737	0.9368
0.19	1.4104	1.2281	0.9174	0.42	1.8048	1.0784	0.9378	0.65	1.8235	1.0737	0.9368
0.20	1.4313	1.223	0.9225	0.43	1.8092	1.0773	0.9376	0.66	1.8235	1.0737	0.9368
0.21	1.451	1.216	0.9267	0.44	1.8126	1.0764	0.9375	0.67	1.8235	1.0737	0.9368
0.22	1.4702	1.2075	0.9301	0.45	1.8153	1.0757	0.9374	≥ 0.68	1.8235	1.0737	0.9368

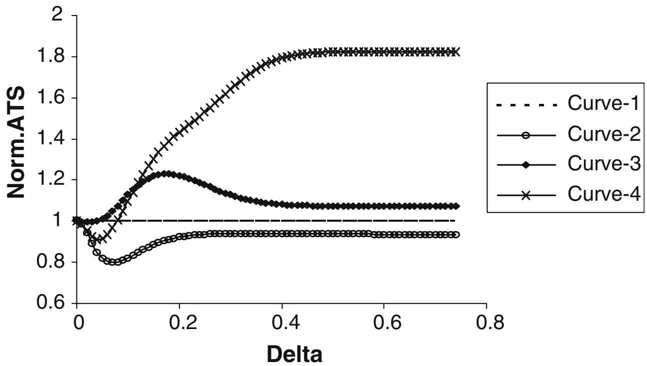

Fig. 1 Graphs of normalised ATS for four charts. The *curves 1, 2, 3* and *4*, respectively, correspond to $ATS_{GR}(\delta)$, $ATS_{SSGR}(\delta)$, $ATS_S(\delta)$ and $ATS_{\overline{X}}(\delta)$

It is to be noted that, in this example, $\delta_1 = 0.2$ and in such situations a very minor shift say not exceeding 0.01 might not matter much, so as to stop the process. Therefore, we conclude that SSGR chart performs better than remaining three charts.

3.2 Example 2

Let

$$\begin{array}{cccc} \delta_1: & 0.2 & 0.5 & 1 \\ \tau: & 2{,}000 & 10{,}000 & 50{,}000. \end{array}$$

Table 3 Showing the optimum values of control parameters and ATS(δ_1) of the four charts

Example	$\overline{X}$ chart			Synthetic chart				GR chart				SSGR chart			
	n	k	ATS_1	n	k	L	ATS_1	n	k	L	ATS_1	n	k	L	ATS_1
(1, 1)	112	1.911	193	95	1.495	3	146	63	1.457	4	124	61	1.29	3	113
(2, 1)	32	2.409	48	19	1.896	3	33	16	1.63	3	27	15	1.55	3	25
(3, 1)	11	2.776	16	6	2.143	3	10	5	1.823	3	8	5	1.74	3	8
(1, 2)	186	2.353	288	102	1.939	4	201	98	1.594	3	164	89	1.52	3	152
(2, 2)	45	2.841	65	25	2.179	3	42	21	1.850	3	34	20	1.77	3	31
(3, 2)	14	3.195	20	8	2.398	3	12	6	2.037	3	10	6	1.95	3	9
(1, 3)	269	2.783	390	149	2.145	3	256	129	1.818	3	205	118	1.74	3	191
(2, 3)	59	3.244	81	31	2.445	3	52	26	2.057	3	40	24	1.98	3	38
(3, 3)	18	3.568	24	10	2.644	3	15	8	2.218	3	11	7	2.15	3	11

Considering all possible nine combinations of the input parameters (δ_1, τ), values of the control parameters along with respective values of ATS(δ_1) are computed for each of the three control charts and are given in Table 3.

In Table 3, in the first column, (i, j) indicates the combination of i^{th} value of δ_1 and j^{th} value of τ. For example, (1,2) indicates ($\delta_1 = 0.2$, $\tau = 10{,}000$).

As these nine cases cover almost all the practical situations, we can conclude the following.

$$n_{\text{ssg}} < n_g \leq n_s \leq n_{\overline{X}},$$

and

$$\text{ATS}_{\text{ssg}}(\delta_1) < \text{ATS}_g(\delta_1) < \text{ATS}_s(\delta_1) < \text{ATS}_{\overline{x}}(\delta_1).$$

Thus, we conclude that, the SSGR chart performs uniformly better than the $\overline{X}$ chart, the synthetic chart and the GR chart.

Remark 1 For the combinations (3, 1) and (3, 3) in Example 2, though the rounded values of ATS_1 for GR chart and SSGR chart are equal, the exact values of ATS_1 for the given combinations are as below.

1. (3, 1): $(\text{ATS}_1)_{\text{GR}} = 8.2038$, $(\text{ATS}_1)_{\text{SSGR}} = 7.6965$
2. (3, 3): $(\text{ATS}_1)_{\text{GR}} = 11.4198$, $(\text{ATS}_1)_{\text{SSGR}} = 10.7783$.

4 Steady state ATS performance of SSGR chart

As the Markov chain representation of the SSGR chart has more than one non-absorbing initial states, it is desirable to study its performance in steady state. The 'Steady State ATS (SSATS) measures average time (in terms of number of units inspected) to signal, when the effect of head start has been faded away. It is nothing but the average of the ATSs corresponding to all possible non-absorbing states in which the process could be. For the purpose, Markov chain representation of the SSGR chart depending on the $\overline{X}$-based procedure (i.e. depending on the status of the group) is discussed subsequently. Table 4 describes classification of various states of the group and their meaning to be used for the Markov chain representation of the SSGR chart.

Table 4 States of the groups and their description

State	Description	State	Description
0	The group being conforming	$\underline{1}$	The non-conformed group with an downward shift in the process mean and the resulting $Y \leq L$
1	The non-conformed group with $Y > L$	$\overline{1}$	The non-conformed group with a upward shift in the process mean and the resulting $Y \leq L$
$\overline{\underline{1}}$	The non-conformed group at time zero with $Y_0 \leq L$ and a shift in the process mean is of both the type		

Table 5 The initial states corresponding to SSGR chart when $L = 2$

Sr. no.	State	Sr. no.	State	Sr. no.	State
1.	$\overline{\underline{1}}$	5.	0010	9.	$\underline{1}0$
2.	$\overline{\underline{1}}0$	6.	$\overline{1}$	10.	Signal
3.	00	7.	$\overline{1}0$		
4.	001	8.	$\underline{1}$		

Though $\overline{\underline{1}}$ is an imaginary state, it is a useful head start. For the illustration purpose assume that $L = 2$. Thus, the SSGR chart will produce a signal if Y_1 is not exceeding 2 or for some $r(> 1)$, Y_r and $Y_{(r+1)}$ are not exceeding 2 for the first time with the r^{th} and $(r+1)^{\text{th}}$ non-conforming groups are having shift of the same type. A Markov chain representation in this situation can be described by using 10 states listed in Table 5.

In Table 5, the state 00 indicates the sequence of at least 2 $(= L)$ conforming groups. Similarly the other states can be described.

For given L, the Markov chain related to SSGR chart has the following non-absorbing states.

1. A sequence starting with $\overline{\underline{1}}$ and further followed by at most $(L - 1)$ zeros. There are L such states. (states 1 and 2 in the above example.)
2. A sequence of at least L zeros. There is one such state. (state 3 in the above example.)
3. A sequence in (2) followed by 1 and further appended by at most $(L - 1)$ zeros. There are L such sequences. (In the above example states 4 and 5.)
4. $\overline{1}$ ($\underline{1}$) is followed by at most $(L - 1)$ zeros. (In the above example the states 6 to 9.)

Thus, the total number of non-absorbing states are $L + 1 + L + 2L = 4L + 1$. Therefore, the matrix R of non-absorbing states is a square matrix of order $4L + 1$. Note that the $(i, j)^{\text{th}}$ element of R is

$$R(i, j) = \begin{cases} \text{Q} & \text{if the } i^{\text{th}} \text{ state leads to } j^{\text{th}} \text{ state, and } j^{\text{th}} \text{ state} \\ & \text{corresponds to the sequence ending with 0} \\ \alpha\text{P} & \text{if the } i^{\text{th}} \text{ state leads to } j^{\text{th}} \text{ state, and } j^{\text{th}} \text{ state is } \overline{1} \\ (1-\alpha)\text{P} & \text{if the } i^{\text{th}} \text{ state leads to } j^{\text{th}} \text{ state, and } j^{\text{th}} \text{ state is } \underline{1}. \\ 0 & \text{otherwise.} \end{cases}$$

Table 6 Values of steady state (ATS) corresponding to various δ values for the four charts considered in example-1

δ	$\overline{X}$chart	Synthetic chart		GR chart		SSGR chart	
	SSATS	SSATS	Adj (SSATS)	SSATS	Adj (SSATS)	SSATS	Adj (SSATS)
0	9,999.825	1,1725.36	10,000	13,531	10,000	16,039	10,000
0.1	1,153.281	1,426.76	1,247.518	1,580.59	1,168.125	1373.4	856.2878
0.2	287.984	284.51	242.645	306.36	226.4134	207.4678	129.3521
0.26	2,10.5349	177.3712	151.2714	204.6359	151.2349	124.646	77.71432
0.3	193.9732	148.52	126.6656	178.62	132.008	105.4238	65.72966
0.4	186.1791	123.3	106.8624	158.91	117.4414	90.8277	56.62928
0.5	186.0007	122.55	104.517	156.89	115.9486	89.1031	55.55402
0.6	186	122.4	104.389	156.8	115.882	89.0026	55.49136
0.7	186	122.4	104.389	156.8	115.882	89	55.48974

Let $\underline{\pi}$ be a $1 \times (4L + 1)$ row vector corresponding to the stationary probability distribution then the elements of $\underline{\pi}$ represent the conditional probabilities that the Markov chain will be in each of the non-absorbing states, conditioned on no signal. As shown in Brook and Evans (1972), the vector of 'Average Run Lengths' $\underline{\text{(ARLs)}}$ is given by $(I - R)^{-1}\ \underline{1}$, where ARL is nothing but the average number of groups inspected by the time the signal is received. Further as mentioned in Davis and Woodall (2002), the steady state ARL is given by $\underline{\pi} \times \underline{\text{ARL}}$ and the SSATS of the SSGR chart is the product of n_{ssg} and the steady state ARL.

Let Y be the group run length. We shall note that, for any positive real numbers s and t, $P\{Y > t + s | Y > s\} = P\{Y > t\}$. Hence, for the synthetic, GR and SSGR chart, the SSATS is not smaller than the zero state ATS. Therefore, the steady state performance of the two charts should be compared by making the $(\text{SSATS})_0$ of the two charts identical. For the purpose, if chart I and chart II are the two control charts to detect shifts in the process mean, we compute the adjusted SATS of the chart II w.r.t. chart I as

$$\left[\text{Adj. SSATS}(\delta)\right]_{\text{II}} = \{[\text{SSATS}(\delta)]_{\text{II}} \,/\, [(\text{SSATS})_0)]_{\text{II}}\}\,\{[(\text{SSATS})_0]_{\text{I}}\}\,. \quad (10)$$

Example 1 (Cont.) Table 6 gives SSATS and the adjusted SSATS values corresponding to various values of δ for all the four charts.

From the Table 6, we conclude that for any type of shift in the process mean, the steady state performance of SSGR is better than the other three charts.

5 Conclusions

Except perhaps very small shifts in the process mean, SSGR control chart proposed here performs significantly better than the $\overline{X}$ chart, the synthetic chart and the GR chart. The $\text{ATS}_{\text{ssg}}(\delta)$ is significantly less than that for the remaining three charts. Hence, we strongly recommend the use of SSGR chart in industries.

It is to be noted that, though the proposed control charts are based on 100% inspection, they can be used to non-100% inspection cases, if the uniform sampling is used. In practice, at each sample level, curtailed sampling scheme can be used to test the current process level.

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